when the two conditions (3. 1) are violated, components of the echo signal image can also be determined by formulas (2. 8) — (2. 10), with function B appearing in these determined by formula

$$B = [2z(1 - z^2)^{1/2} \sin^{-1} \vartheta]^{1/2} \tag{3.3}$$

although the application of formula (3. 3) is justified only for considerable values of z and $\sin \vartheta$.

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ON A PARTICULAR CLASS OF SOLUTIONS OF TRIPLE INTEGRAL EQUATIONS

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A new class of solutions of triple integral equations is proposed. A number of boundary value problems of the elasticity theory with mixed boundary conditions (problems of contact, cracks, etc.) can be reduced to this class.

1. Let us consider triple integral equations of the form

$$\int_0^\infty \Phi(\xi) J_\nu(\xi x)\, d\xi = G_1(x) \quad (0 < x < a) \tag{1.1}$$

$$\int_0^\infty \xi^{-2\alpha} \Phi(\xi) J_\nu(\xi x)\, d\xi = F_2(x) \quad (a < x < b)$$

$$\int_0^\infty \Phi(\xi) J_\nu(\xi x)\, d\xi = G_3(x) \quad (b < x < \infty)$$

where functions G_1, F_2 and G_3 are assumed known, Φ is the unknown function, and $J_\nu(x)$ is a Bessel function of the first kind.

The most important results concerning the solution of the system of Eqs. (1. 1) appear in [1 — 6].

Setting [6]

$$\Phi(\xi) = \xi\psi(\xi), \quad f(x) = \left(\frac{2}{x}\right)^{2\alpha} F(x), \quad g(x) = G(x) \tag{1. 2}$$

and using the Hankel operator

$$S_{\eta,\alpha} f(x) = 2^{\alpha} x^{-\alpha} \int_0^{\infty} t^{1-\alpha} J_{2\eta+\alpha}(xt) f(t)\, dt$$

the triple equations (1. 1) may be written in the compact form

$$S_{\nu/2-\alpha, 2\alpha}\psi(x) = f(x), \quad S_{\nu/2, 0}\psi(x) = g(x) \tag{1. 3}$$

where

$$f(x) = \begin{cases} f_1(x) & (0 < x < a) \\ f_2(x) & (a < x < b) \\ f_3(x) & (b < x < \infty) \end{cases} \tag{1. 4}$$

Function $g(x)$ can be similarly represented.

The triple equations (1. 1) (or (1. 3)) can be reduced to a system of two integral equations or to one Fredholm's integral equation of the second kind. Papers [1, 3, 6] deal with this question on the basis of Seddon's "trial" solution [7]

$$\psi(x) = S_{\nu/2, -\alpha} h(x) \tag{1. 5}$$

The solutions in [1, 3] were originally obtained without resorting to the Erdélyi-Kober and Hankel operators which made them fairly cumbersome. To simplify the analysis the use of Hankel and Erdélyi-Kober operators was suggested in [7, 8]. Solutions with the use of these operators presented in [1, 3], appear in their compact form in [6].

The Erdélyi-Kober operators are defined as follows:

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2 - u^2)^{\alpha-1} u^{2\eta+1} f(u)\, du \quad (\alpha > 0) \tag{1. 6}$$

$$I_{\eta,\alpha} f(x) = \frac{x^{-2\eta-2\alpha-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_0^x (x^2 - u^2)^{\alpha} u^{2\eta+1} f(u)\, du \quad (-1 < \alpha < 0)$$

$$K_{\eta,\alpha} f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^{\infty} (u^2 - x^2)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u)\, du \quad (\alpha > 0) \tag{1. 7}$$

$$K_{\eta,\alpha} f(x) = -\frac{x^{2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_x^{\infty} (u^2 - x^2)^{\alpha} u^{-2\alpha-2\eta+1} f(u)\, du \quad (-1 < \alpha < 0)$$

where $\Gamma(x)$ is the gamma function.

Seddon had shown [8, 9] that

$$S_{\eta+\alpha, \beta} S_{\eta,\alpha} = I_{\eta, \alpha+\beta}, \quad S_{\eta,\alpha} S_{\eta+\alpha, \beta} = K_{\eta, \alpha+\beta} \tag{1. 8}$$

2. A new class of solutions of the system of Eqs. (1. 3) can be obtained by setting

$$\psi(x) = S_{\nu/2+\alpha, -\alpha} H(x) \tag{2. 1}$$

instead of the trial solution (1. 5). Substituting (2. 1) into (1. 3), with the use of (1. 8) we obtain

$$K_{\nu/2-\alpha,\alpha}H = f, \quad I_{\nu/2+\alpha,-\alpha}H = g \tag{2.2}$$

Solution of these equations yields

$$H = K^{-1}_{\nu/2-\alpha,\alpha}f, \quad H = I^{-1}_{\nu/2+\alpha,-\alpha}g \tag{2.3}$$

where $I^{-1}_{\eta,\alpha}$ and $K^{-1}_{\eta,\alpha}$ are inverse operators. It is shown in [8, 9] that

$$I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha}, \quad K^{-1}_{\eta,\alpha} = K_{\eta+\alpha,-\alpha} \tag{2.4}$$

Using the substitution (2. 1) we obtain solutions which to a certain extent are "parallel" to those presented in [1, 3, 6]. As done by the authors of those papers, we set

$$g_1 = 0, \quad g_3 = 0 \tag{2.5}$$

and shall, furthermore, assume that $-1 < \alpha < 1$.

It follows from Eqs. (2. 2) and (2. 3), and relationships (2. 5), that

$$\begin{aligned}
H_1 &= 0 \\
H_2 &= \binom{b}{x} K^{-1}_{\nu/2-\alpha,\alpha}f_2 + \binom{\infty}{b} K^{-1}_{\nu/2-\alpha,\alpha}f_3 \\
H_3 &= \binom{b}{a} I^{-1}_{\nu/2+\alpha,-\alpha}g_2 \\
f_3 &= \binom{\infty}{x} K_{\nu/2-\alpha,\alpha}H_3, \quad g_2 = \binom{x}{a} I_{\nu/2+\alpha,-\alpha}H_2 \\
f_1 &= \binom{b}{a} K_{\nu/2-\alpha,\alpha}H_2 + \binom{\infty}{b} K_{\nu/2-\alpha,\alpha}H_3
\end{aligned} \tag{2.6}$$

where the letters in parentheses in front of operators represent the new integration limits.

In what follows it is expedient to use operators L and M which were introduced by Cooke [6]

$$\begin{aligned}
&\binom{d}{c} I^{-1}_{\eta,\alpha} \binom{x}{c} I_{\eta,\alpha}f(x) = -\begin{pmatrix} x, d \\ d, c \end{pmatrix} L_{\eta,\alpha}f(x) \quad (x > d > c) \\
&\binom{e}{d} K^{-1}_{\eta,\alpha} \binom{e}{x} K_{\eta,\alpha}f(x) = -\begin{pmatrix} d, e \\ x, d \end{pmatrix} M_{\eta,\alpha}f(x) \quad (x < d < e)
\end{aligned} \tag{2.7}$$

Substituting in (2. 6) its fourth formula into the second and the fifth into the third, and taking into account formulas (2. 4) and (2. 7), we obtain the system of equations

$$\begin{aligned}
H_2 &= \binom{b}{x} K^{-1}_{\nu/2-\alpha,\alpha}f_2 - \begin{pmatrix} b, \infty \\ x, b \end{pmatrix} M_{\nu/2-\alpha,\alpha}H_3 \quad (a < x < b) \\
H_3 &= -\begin{pmatrix} x, b \\ b, a \end{pmatrix} L_{\nu/2+\alpha,-\alpha}H_2 \quad (b < x < \infty)
\end{aligned} \tag{2.8}$$

Cooke had shown [6] that

$$\begin{pmatrix} x, f \\ d, e \end{pmatrix} L_{\eta,\alpha}f(x) = \frac{2\sin\alpha\pi}{\pi} x^{-2\eta}(x^2 - d^2)^{-\alpha}\int_e^f \frac{(d^2 - t^2)^{\alpha} t^{2\eta+1} f(t)}{x^2 - t^2}\,dt$$

$$(x > d \geqslant f > e)$$

$$\begin{pmatrix} d, f \\ x, e \end{pmatrix} M_{\eta,\alpha}f(x) = \frac{2\sin\alpha\pi}{\pi} x^{2\eta+2\alpha}(d^2 - x^2)^{-\alpha}\int_e^f \frac{(t^2 - d^2)^{\alpha} t^{-2\alpha-2\eta+1} f(t)}{t^2 - x^2}\,dt$$

$$(x < d \leqslant e < f)$$

It will be seen from these formulas that (2. 8) is a system of integral equations. By solving (2. 8) we can determine functions $H_2(x)$ and $H_3(x)$, while (2. 6) implies that $H_1(x) = 0$. Function $H(x)$ is thus determined, since its representation is similar to (1. 4).

Finally, using formulas (2. 1) it is possible to determine function $\psi(x)$, i. e. to obtain the solution of the system of treble integral equations (1. 3) and, consequently, also that of the input system (1. 1). Most frequently it is not function $\psi(x)$ but functions $f_1(x)$, $f_3(x)$ and $g_2(x)$ that are required. The latter can be obtained by using the last three formulas of (2. 6).

Substituting in (2. 8) the second of its formulas into the first, we obtain for function $H_2(x)$ the integral equation of the second kind

$$H_2(x) = \varphi(x) - \left(\frac{2}{\pi}\right)^2 \int_a^b K(x, y) H_2(y)\, dy \tag{2.9}$$

where

$$\varphi(x) = \binom{b}{x} K_{\nu/2, -\alpha} f_2 \tag{2.10}$$

$$K(x, y) = \sin^2 \alpha\pi \frac{x^\nu y^{1+\nu+2\alpha}}{(b^2 - x^2)^\alpha (b^2 - y^2)^\alpha} \int_b^\infty \frac{t^{1-2\nu-2\alpha} (t^2 - b^2)^{2\alpha}}{(t^2 - x^2)(t^2 - y^2)}\, dt$$

$$(-1/2 < \alpha < 1)$$

Function $\varphi(x)$ can be similarly determined with the use of (1. 7).

We have, thus, derived solutions that are parallel to those obtained in [1, 3, 6].

Thus, for example, in the particular case of $\nu = 0$ and $\alpha = 1/2$ formulas (2. 10) assume the form

$$\varphi(x) = -\frac{2}{\pi^{1/2} x} \frac{d}{dx} \int_x^b \frac{u^2 f_2(u)\, du}{(u^2 - x^2)^{1/2}} \tag{2.11}$$

$$K(x, y) = \frac{y^2}{2(b^2 - x^2)^{1/2} (b^2 - y^2)^{1/2} (x^2 - y^2)} \times$$

$$\left(\frac{b^2 - y^2}{y} \ln \frac{b + y}{b - y} - \frac{b^2 - x^2}{x} \ln \frac{b + x}{b - x}\right)$$

3. To show the difference between the solution (based on the substitution (2. 1)) obtained here and that of Cooke (based on Seddon's substitution (1. 5)) in [1, 6], we consider, as an example, the problem of pressing a ring-shaped punch with a flat base into an elastic half-space. The punch is subjected to a vertical force P acting along its axis of symmetry. It is assumed that outside of the punch the half-space surface is free from stress and that friction between the punch and the half-space is absent. In such case $\nu = 0$ and $\alpha = 1/2$, and furthermore

$$g_1(r) = 0, \quad g_3(r) = 0, \quad f_2(r) = -2\mu\delta/(1 - \nu_0) r \tag{3.1}$$

$$g_2(r) = \tau_z(r, 0) \quad \text{for} \quad a < r < b$$

where μ is the shear modulus, ν_0 is the Poisson ratio of the half-space material, δ is the depth of punch impression, and a and b are the inner and outer diameters of the punch, respectively.

The solution based on the substitution (2. 1) leads in this case, after suitable transformations, to the following equations:

$$\sigma_z(r,0) = -\frac{\gamma_a P}{2\pi r}\frac{d}{dr}\int_\varepsilon^{r/b}\left(\frac{1-y^2}{r^2/b^2-y^2}\right)^{1/2}\psi(y)\,dy \quad (a<r<b) \tag{3.2}$$

$$\frac{1-x^2}{x^2}\psi(x) = 1-\left(\frac{2}{\pi}\right)^2\int_\varepsilon^1 K_a(x,y)\,\psi(y)\,dy$$

$$K_a(x,y) = \frac{1}{2(x^2-y^2)}\left(\frac{1-y^2}{y}\ln\frac{1+y}{1-y} - \frac{1-x^2}{x}\ln\frac{1+x}{1-x}\right)$$

$$\varepsilon = \frac{a}{b}, \quad \gamma_a^{-1} = \int_\varepsilon^1 \psi(y)\,dy, \quad \delta = \gamma_a\frac{P(1-\nu_0)}{4\mu b}$$

where $\sigma_z(r,0)$ is the normal stress in a small area of contact. Formulas (2.6), (2.9), (2.11) and (3.1) were used for deriving Eqs. (3.2).

The solution of the same problem based on Seddon's substitution (1.5) and results obtained by Cooke [1, 6] yield equations [10]

$$\sigma_z(r,0) = \frac{\gamma_b P}{2\pi r}\frac{d}{dr}\int_{r/b}^{1}\left(\frac{y^2-\varepsilon^2}{y^2-r^2/b^2}\right)^{1/2}\eta(y)\,dy \quad (a<r<b) \tag{3.3}$$

$$\frac{x^2-\varepsilon^2}{x^2}\eta(x) = 1-\left(\frac{2}{\pi}\right)^2\int_\varepsilon^1 K_b(x,y)\,\eta(y)\,dy$$

$$K_b(x,y) = \frac{1}{2(x^2-y^2)}\left(\frac{x^2-\varepsilon^2}{x}\ln\frac{x+\varepsilon}{x-\varepsilon} - \frac{y^2-\varepsilon^2}{y}\ln\frac{y+\varepsilon}{y-\varepsilon}\right)$$

$$\gamma_b^{-1} = \int_\varepsilon^1 \eta(y)\,dy, \quad \delta = \gamma_b\frac{P(1-\nu_0)}{4\mu b}$$

It can be readily shown that $\gamma_a = \gamma_b = \gamma$.

The following conclusions can be drawn from the analysis and comparison of formulas (3.2) and (3.3).

a) The solution of the problem is completely determined by formulas (3.2) or (3.3).

b) Formulas (3.2) make it possible to obtain the asymptotic representation for $\sigma_z(r,0)$ when $r\to a+0$, i.e. when approaching the punch inner contour, while formulas (3.3) yield that stress for $r\to b-0$, i.e. when approaching the punch outer contour (for further details of this see [11]). Thus solutions (3.2) and (3.3) are in a way complementary.

c) The kernel $K_a(x,y)$ of Fredholm's integral equation of the second kind is independent of parameter $\varepsilon = a/b$, which may facilitate computations (and simplify computer programing), when the problem is to be solved for various values of ε. In solution (3.3) the kernel $K_b(x,y)$ depends on ε.

d) When $\varepsilon = 0$ the solutions of integral equations (3.2) and (3.3) are of the form

$$\psi(x) = \frac{x}{\pi(1-x^2)^{1/2}}\ln\frac{1+x}{1-x}, \quad \eta(x) = 1 \tag{3.4}$$

Solutions of integral equations (3.2) and (3.3) were derived for $\varepsilon \neq 0$ by numerical methods. The integrals in these equations were replaced by Gauss' quadrature formula (the number of nodes was nearly 40). The system of linear algebraic equations obtained in this way was solved on a computer. The computation results (for $\varepsilon \neq 0$) and formulas (3.4) (for $\varepsilon = 0$) were used for plotting the curves of functions $\psi(x)$ and $\eta(x)$ for several values of ε. These curves are represented in Fig. 1 by solid and dash lines, respectively. It is seen that the curves of function $\eta(x)$ related to different values of ε differ considerably between themselves, while for $(0 \leqslant \varepsilon \leqslant 0.5)$ the curves of $\psi(x)$ are virtually the same. This property may be used for the derivation of approximate solutions for $\varepsilon \leqslant 0.5$. In the considered case the substitution of the expression for $\psi(x)$ from (3.4) into the first formula in (3.2) yields the approximate formula for computing the normal stress at the small area of contact between the ring-shaped punch and the half-space (for $\varepsilon \leqslant 0.5$).

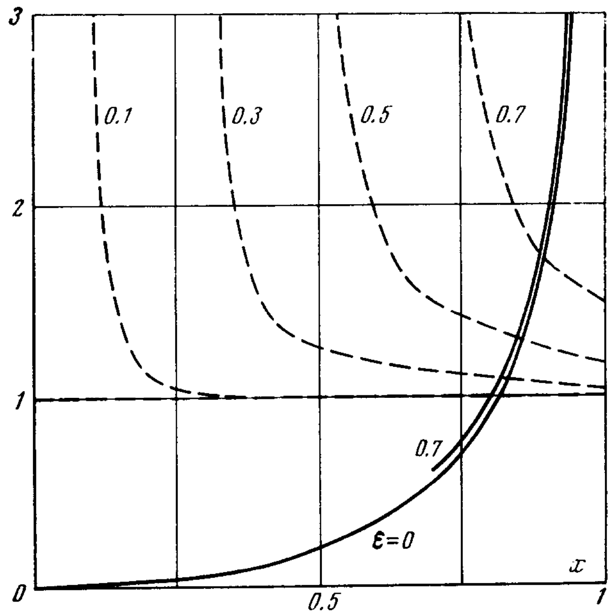

Fig. 1

4. Finally, we shall show how solutions (1.5) and (2.1) can be obtained. We seek the solution of Eqs. (1.3) of the form

$$\psi = S_{\beta,\gamma}\xi$$

Two cases are possible here.

1°. Equations (1.3) may be reduced to the form

$$I_{\mu_1,\lambda_1}\xi = f, \quad K_{\mu_2,\lambda_2}\xi = g$$

These equations occur when

$$S_{\nu/2-\alpha,2\alpha}S_{\beta,\gamma} = I_{\mu_1,\lambda_1}, \quad S_{\nu/2,0}S_{\beta,\gamma} = K_{\mu_2,\lambda_2} \tag{4.1}$$

Using (1.8) and (4.1) we obtain the relationships

$$\beta + \gamma = \nu/2 - \alpha, \quad \mu_1 = \beta, \quad \lambda_1 = 2\alpha + \gamma$$
$$\beta = \nu/2, \quad \mu_2 = \nu/2, \quad \lambda_2 = \gamma$$

which yield

$$\beta = \nu / 2, \ \gamma = -\alpha, \ \mu_1 = \nu / 2, \ \mu_2 = \nu / 2, \ \lambda_1 = \alpha, \ \lambda_2 = -\alpha$$

Hence in this case $\psi = S_{\nu/2, -\alpha}\xi$, i.e. we have Sneddon's solution (1.5).

2°. Equations (1.3) can be reduced to the form

$$K_{\mu_3, \lambda_3}\xi = f, \quad I_{\mu_4, \lambda_4}\xi = g$$

Further computations are carried out by the same plan as in the first case. As the result we have

$$\beta = \nu / 2 + \alpha, \ \gamma = -\alpha, \ \mu_3 = \nu / 2 - \alpha, \ \mu_4 = \nu / 2 + \alpha, \ \lambda_3 = \alpha, \ \lambda_4 = -\alpha$$

In this case $\psi = S_{\nu/2+\alpha, -\alpha}\xi$, i.e. we obtain solution (2.1).

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